Eur. Phys. J. D **18**, 129–135 (2002)
DOI: 10.1140/e10053-002-0014-7

THE EUROPEAN PHYSICAL JOURNAL D
EDP Sciences

Collective atomic spin squeezing and control

L. Vernac, M. Pinard[a], V. Josse, and E. Giacobino

Laboratoire Kastler Brossel, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris Cedex 05, France

Received 14 August 2001 and Received in final form 7 November 2001

Abstract. Using a quantum theory for an ensemble of two- or three-level atoms driven by electromagnetic fields in an optical cavity, we show that the various spins associated with the atomic ensemble can be squeezed. Two kinds of squeezing are obtained: on the one hand self-spin squeezing when the input fields are coherent ones and the atomic ensemble exhibits a large non-linearity; on the other hand squeezing transfer when one of the incoming fields is squeezed.

PACS. 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps – 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements

1 Introduction

The possibility of engineering the quantum state of atomic ensembles raises a lot of interest in connection with the needs of quantum information processing based on continuous variables. Atomic ensembles could be either sources or registers for quantum variables that are transmitted by electromagnetic fields [1–3]. In this paper, we present calculations performed on model systems consisting of two- or three-level atoms that interact respectively with one or two fields. The atoms are placed in an optical cavity which ensures a sizeable level of interaction with the field while keeping the absorption rather low, in contrast with single pass schemes. In such conditions, the validity of quantum fluctuations calculations based on quantum Langevin equations is well established.

We will study two regimes. In the first one, the atoms interact with coherent light. One can achieve atomic self-spin squeezing if the non-linearity of the medium is high enough. This atomic squeezing originates from the same physical process as the squeezing of the field going out of the cavity, which was theoretically predicted in references [4–6] and experimentally observed a little later [7–9]. Already in the early 1980's, it was conjectured that atomic spin squeezing appeared as a counterpart of squeezing of the electromagnetic field [10,11]. However, the quantum noise reduction on atomic variables computed in several papers can be obtained by a mere rotation of the atomic variables of a two-level system interacting with a coherent field. As shown in references [12–14], in order to be relevant, spin squeezing should be computed in the plane orthogonal to the direction of the mean spin, where the mean value of the spin component is zero. We have performed full quantum calculations in the relevant basis. Here we generalize the results obtained previously on two-level systems [15] to the more practical case of three-level systems.

In the second regime, the atoms interact with a squeezed incoming field for the two level system, with one coherent field and one squeezed field for the three-level system. This configuration has already been used in references [16–19] to study the modification of the spontaneous emission due to the interaction of atoms with squeezed light. However, the fact that the field squeezing can be transferred to the atomic system was only studied later. This scheme was recently proposed and experimentally tested in a single pass configuration to produce spin squeezing [20,21]. The cavity configuration that we propose here should allow to enhance the squeezing transfer between atoms and field while minimizing the interaction with the surrounding vacuum field fluctuations. We find that the optimal conditions for the transfer of squeezing from the incident light to the atoms correspond to a non saturating intensity, in the strong coupling regime.

2 Equations for atomic fluctuations

2.1 Two-level atoms interacting with one field in an optical cavity

We first consider an ensemble of N motionless two-level atoms placed inside a high-finesse single-ended optical cavity and interacting with a single mode field. The round-trip time in the cavity is τ, the decay rate of the field in the cavity is κ. The atomic system has a ground state g and an excited state e, separated by the energy $\hbar\omega_0$. We call γ the decay rate of the atomic dipole, due to a purely radiative process. The atoms are driven by a field the

[a] e-mail: pinard@spectro.jussieu.fr

frequency of which is ω_L. This field is represented by the operator $A(t)e^{-i\omega_L t}$. The mean square value of the field will be expressed in number of photons per second. The cavity resonance closest to ω_L has a frequency ω_C. We define the atomic and cavity detuning parameters as $\Delta = (\omega_0 - \omega_L)$ and $\Delta_C = (\omega_C - \omega_L)$. The atom-field coupling constant is $g = \mathcal{E}_0 d/\hbar$, where d is the atomic dipole, and $\mathcal{E}_0 = \sqrt{\hbar\omega_L/2\epsilon_0 \mathcal{S} c}$.

We define the collective polarization $P(t)$ and the collective population difference $S_z(t)$ as:

$$P(t) = \sum_{i=1}^{N} \sigma_i(t), \quad S_z(t) = \sum_{i=1}^{N} \sigma_{zi}(t) \tag{1}$$

with $\sigma_i(t)$ is the lowering operator for individual atoms in the rotating frame, $\sigma_i(t) = |g_i\rangle\langle e_i| e^{+i\omega_L t}$ and $\sigma_{zi}(t) = (|e_i\rangle\langle e_i| - |g_i\rangle\langle g_i|)/2$.

The field inside the cavity is related to the incident field and to the atomic polarization by

$$\frac{dA(t)}{dt} = -(\kappa + i\Delta_C)A(t) + i(g/\tau)P(t) + \sqrt{2\kappa/\tau}A^{in}(t). \tag{2}$$

The variation of the intracavity field is due to the recycling of the field of the cavity and loss through the coupling mirror, to the field emitted by the atomic polarization and to the incoming field A^{in}. The fluctuations of the incoming field can be seen as a Langevin force for this equation. The atomic polarization and populations are given by quantum Langevin equations derived from the Bloch equations by adding the Langevin forces corresponding to the coupling with the vacuum field surrounding the system

$$\frac{dP(t)}{dt} = -(\gamma + i\Delta)P(t) - 2igA(t)S_z(t) + F_P(t), \tag{3}$$

$$\frac{dS_z(t)}{dt} = -2\gamma(S_z(t) + N/2) - ig\left(A^\dagger(t)P(t) - A(t)P^\dagger(t)\right) + F_{S_z}(t). \tag{4}$$

The noise operators $F_P(t)$, $F_{P^\dagger}(t)$ and $F_{S_z}(t)$ are characterized by zero averages and by correlation functions that are given in the Appendix.

We will be interested in the quantum fluctuations of the field and atomic operators around their steady states mean values such as $A(t) = a_0 + \delta A(t)$, $A^{in}(t) = a_{in} + \delta A^{in}(t)$, $P(t) = p_0 + \delta P(t)$, $S_z(t) = s_{z0} + \delta S_z(t)$, where a_{in}, a_0, p_0, and s_{z0} are the steady state mean values. We choose the phases such that a_0 is real.

The mean values can easily be computed from equations (2–4) without the fluctuating terms, which are the usual Maxwell-Bloch equations.

The steady state solution of this system is given by

$$p_0 = \frac{iN\beta_0(1 - i\delta)}{1 + \delta^2 + 2|\beta_0|^2} \tag{5}$$

$$s_{z0} = -\frac{1 + \delta^2}{2(1 + \delta^2 + 2|\beta_0|^2)} \tag{6}$$

$$\sqrt{2/\kappa\tau}\beta_{in} = \beta_0\left[\left(1 + \frac{2C}{1 + \delta^2 + 2|\beta_0|^2}\right) + i\left(\delta_C - \frac{2C\delta}{1 + \delta^2 + 2|\beta_0|^2}\right)\right] \tag{7}$$

where we have introduced the scaled fields $\beta_0 = ga_0/\gamma$, $\beta_{in} = ga_{in}/\gamma$, the scaled detunings $\delta = \Delta/\gamma$ and $\delta_c = \Delta_c/\kappa$ and the cooperativity parameter $C = Ng^2/2\kappa\gamma\tau$. For large enough values of β_0 and C, the solution exhibits the well known bistable behavior of the intra-cavity field. It is in the vicinity of the bistable turning point that the squeezing of the outgoing field is maximum, and we will look for self-spin squeezing in the same range of parameters. On the other hand, for large g (or large C), one can reach, even for small β_0, the regime of vacuum Rabi splitting or strong coupling regime.

To obtain equations for the field and atomic fluctuations, we linearize equations (2–4)

$$\frac{d\delta A(t)}{dt} = -(\kappa + i\Delta_C)\delta A(t) + i(g/\tau)\delta P(t) + \sqrt{2\kappa/\tau}\delta A^{in}(t), \tag{8}$$

$$\frac{d\delta P(t)}{dt} = -(\gamma + i\Delta)\delta P(t) - 2iga_0\delta S_z(t) - 2ig\delta A(t)s_{z0} + F_P(t), \tag{9}$$

$$\frac{d\delta S_z(t)}{dt} = -2\gamma\delta S_z(t) - iga_0\left(\delta P(t) - \delta P^\dagger(t)\right) - ig\left(p_0\delta A^\dagger(t) - p_0^*\delta A(t)\right) + F_{S_z}(t). \tag{10}$$

In order to evaluate the spin squeezing, we will compute the variances of the spin components that can be calculated in the following way. We can write equations (8–10) and the hermitian conjugates of equations (8, 9) in a matrix form as

$$\frac{d\,|\delta\xi(t)]}{dt} = -[B]\,|\delta\xi(t)] + |F_\xi] \tag{11}$$

where $|\delta\xi(t)]$ is a column vector

$$|\delta\xi(t)] = \left[\delta A(t), \delta A^\dagger(t), \delta P(t), \delta P^\dagger(t), \delta S_z(t)\right|^T \tag{12}$$

$[B]$ is the linearized evolution matrix of the atom-field system and $|F_\xi]$ is the column vector

$$|F_\xi(t)] = \left[\sqrt{2\kappa/\tau}\delta A^{in}(t), \sqrt{2\kappa/\tau}\delta A^{in\dagger}(t), F_P(t), F_{P^\dagger}(t), F_{S_z}(t)\right|^T. \tag{13}$$

We define the covariance matrix $[G(t)]$ by

$$[G(t)] = |\delta\xi(t)]\,[\delta\xi(0)| \tag{14}$$

and the diffusion matrix by

$$|F_\xi(t)]\,[F_\xi(t')| = [D]\,\delta(t-t'). \quad (15)$$

The value of the diffusion matrix $[D]$ is given in the Appendix for the case of a broadband squeezed input field.

A single mode squeezed input field can be written as [23]:

$$\hat{a}_s = \hat{a}\cosh(r) - \hat{a}^\dagger \mathrm{e}^{\mathrm{i}\theta}\sinh(r) \quad (16)$$

$$\hat{a}_s^\dagger = \hat{a}^\dagger\cosh(r) - \hat{a}\mathrm{e}^{-\mathrm{i}\theta}\sinh(r) \quad (17)$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the operators describing a coherent field, θ gives the phase of the squeezing, and r is the parameter so that the amount of squeezing is e^{-2r}. This formula can be generalized to a broadband input field [24], the correlation functions of which are given in the appendix.

The variances of the spin components and their correlation functions are the elements of the zero time correlation matrix $[G(0)]$, which verifies [11]:

$$[B]\,[G(0)] + [G(0)]\,[B]^\dagger = [D]\,. \quad (18)$$

Inverting equation (18) gives $[G(0)]$ and consequently the spin variances.

2.2 Three-level atoms interacting with two fields in an optical cavity

In a two-level system, it is difficult to probe the fluctuations of the atomic dipole independently of the driving field. A three-level system is better suited to the atomic fluctuations measurements. We consider three levels labeled 0, 1, 2 in a V-configuration. The atoms interact with two light fields, a "pump" A_p, with frequency ω_p and a "probe" A_s, with frequency ω_s in an optical cavity, respectively close to resonance with transitions $0 \to 1$ and $0 \to 2$. The detunings from atomic resonance are $\Delta_1 = \omega_{01} - \omega_\mathrm{p}$ for the pump, and $\Delta_2 = \omega_{02} - \omega_\mathrm{s}$ for the probe. The cavity resonance frequencies the closest to the pump and probe frequencies are respectively $\omega_{\mathrm{c}1}$ and $\omega_{\mathrm{c}2}$. The cavity detunings for the pump and probe fields are $\Delta_{\mathrm{c}1} = \omega_\mathrm{p} - \omega_{\mathrm{c}1}$ et $\Delta_{\mathrm{c}2} = \omega_\mathrm{s} - \omega_{\mathrm{c}2}$. The incoming fields are A_p^in and A_s^in.

The three-level system is described using 9 collective operators for the N atoms of the ensemble, the populations of levels $|0\rangle$, $|1\rangle$ et $|2\rangle$:

$$\Pi_0 = \sum_{i=1}^N |0\rangle_i \langle 0|_i\,,\;\; \Pi_1 = \sum_{i=1}^N |1\rangle_i \langle 1|_i\,,\;\; \Pi_2 = \sum_{i=1}^N |2\rangle_i \langle 2|_i \quad (19)$$

the components of the optical dipoles in the frames rotating at the frequency of their corresponding lasers and their Hermitian conjugates

$$P_1(t) = \sum_{i=1}^N |0\rangle_i \langle 1|_i\, \mathrm{e}^{\mathrm{i}\omega_\mathrm{p} t},\;\; P_2(t) = \sum_{i=1}^N |0\rangle_i \langle 2|_i\, \mathrm{e}^{\mathrm{i}\omega_\mathrm{s} t} \quad (20)$$

and the components of the dipole associated to the coherence between levels $|1\rangle$ and $|2\rangle$ (and its Hermitian conjugate):

$$P_r(t) = \sum_{i=1}^N |2\rangle_i \langle 1|_i\, \mathrm{e}^{\mathrm{i}(\omega_\mathrm{p}-\omega_\mathrm{s})t}. \quad (21)$$

The coupling constants between atoms and fields, defined as previously are denoted g_p for the pump and g_s for the probe. The decay constant of dipoles P_1 and P_2 are respectively γ_1 and γ_2. Let us note that the system we describe is a closed one and the operator sum of the populations is proportional to I_d the identity operator:

$$\Pi_0(t) + \Pi_1(t) + \Pi_2(t) = N I_\mathrm{d}. \quad (22)$$

The atomic state is always an eigenstate of $\Pi_0(t)+\Pi_1(t)+\Pi_2(t)$, and no noise is associated to this operator. The evolution of the atomic system can be studied in an 8-dimension space by defining:

$$S_{z1}(t) = \frac{1}{2}(\Pi_1(t) - \Pi_0(t)), S_{z2}(t) = \frac{1}{2}(\Pi_2(t) - \Pi_0(t)). \quad (23)$$

The Heisenberg-Langevin equations for the atomic variables are obtained in the same way as for the two level system

$$\begin{aligned}\frac{\mathrm{d}S_{z1}}{\mathrm{d}t} &= -(2\gamma_1+\gamma_2)\frac{N}{3} - \frac{2}{3}(4\gamma_1-\gamma_2)S_{z1}\\ &\quad -\frac{4}{3}(\gamma_2-\gamma_1)S_{z2} - \left(\mathrm{i}g_\mathrm{p}A_\mathrm{p}^\dagger P_1 - \mathrm{i}g_\mathrm{p}A_\mathrm{p}P_1^\dagger\right)\\ &\quad -\frac{1}{2}(\mathrm{i}g_\mathrm{s}A_\mathrm{s}^\dagger P_2 - \mathrm{i}g_\mathrm{s}A_\mathrm{s}P_2^\dagger) + F_{S_{z1}}(t), \quad (24)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}S_{z2}}{\mathrm{d}t} &= -(2\gamma_2+\gamma_1)\frac{N}{3} - \frac{2}{3}(4\gamma_2-\gamma_1)S_{z2}\\ &\quad -\frac{4}{3}(\gamma_1-\gamma_2)S_{z1} - \frac{1}{2}\left(\mathrm{i}g_\mathrm{p}A_\mathrm{p}^\dagger P_1 - \mathrm{i}g_\mathrm{p}A_\mathrm{p}P_1^\dagger\right)\\ &\quad -(\mathrm{i}g_\mathrm{s}A_\mathrm{s}^\dagger P_2 - \mathrm{i}g_\mathrm{s}A_\mathrm{s}P_2^\dagger) + F_{S_{z2}}(t), \quad (25)\end{aligned}$$

$$\frac{\mathrm{d}P_1}{\mathrm{d}t} = -(\mathrm{i}\Delta_1+\gamma_1)P_1 - 2\mathrm{i}g_\mathrm{p}A_\mathrm{p}S_{z1} - \mathrm{i}g_\mathrm{s}A_\mathrm{s}P_r + F_{P_1}(t), \quad (26)$$

$$\frac{\mathrm{d}P_2}{\mathrm{d}t} = -(\mathrm{i}\Delta_2+\gamma_2)P_2 - 2\mathrm{i}g_\mathrm{s}\hat{A}_\mathrm{s}S_{z2} - \mathrm{i}g_\mathrm{p}\hat{A}_\mathrm{p}P_r^\dagger + F_{P_2}(t), \quad (27)$$

$$\begin{aligned}\frac{\mathrm{d}P_r}{\mathrm{d}t} &= -\left(\mathrm{i}(\Delta_1-\Delta_2)+\gamma_1+\gamma_2\right)P_r + \mathrm{i}g_\mathrm{p}A_\mathrm{p}P_2^\dagger\\ &\quad -\mathrm{i}g_\mathrm{s}A_\mathrm{s}^\dagger P_1 + F_{P_r}(t). \quad (28)\end{aligned}$$

The round trip time of the fields in the empty cavity is τ and the decay constant of the two fields in the cavity is assumed to be the same for the two fields and equal to κ. The evolution equations for the fields in the cavity are similar to equation (2):

$$\frac{\mathrm{d}A_\mathrm{p}}{\mathrm{d}t} = -(\kappa+\mathrm{i}\Delta_{\mathrm{c}1})A_\mathrm{p} + \frac{\mathrm{i}g_\mathrm{p}}{\tau}P_1 + \sqrt{\frac{2\kappa}{\tau}}A_\mathrm{p}^\mathrm{in}, \quad (29)$$

$$\frac{\mathrm{d}A_\mathrm{s}}{\mathrm{d}t} = -(\kappa+\mathrm{i}\Delta_{\mathrm{c}2})A_\mathrm{s} + \frac{\mathrm{i}g_\mathrm{s}}{\tau}P_2 + \sqrt{\frac{2\kappa}{\tau}}A_\mathrm{s}^\mathrm{in}. \quad (30)$$

The steady state values can be calculated for the three-level system [8] in the same way as for the two level system, although analytical expressions are quite complicated and difficult to use. We have concentrated on the case in which one of the two fields (the pump) is intense, while the other one (the probe) is well below saturation. For the pump field the system is close to the bistability regime, while the other one is a perturbation. We have verified the stability of the system for all the considered parameters.

Our aim here is to obtain the fluctuations of the spin operators associated with levels 1 and 2. For this, we need to determine the fluctuations of operators $P_r, P_r^\dagger, \Pi_1$ and Π_2 (or S_{z1} and S_{z2}). We will also compute the fluctuations of the spins associated to transitions 0–1 and 0–2, in the same conditions to get a better physical insight into the problem.

As previously, we define a column vector representing the fluctuations of the field and atomic variables. In this case, it is a 12-dimension vector:

$$|\delta\xi(t)] = |\delta A_{\rm p}(t), \delta A_{\rm p}^\dagger(t), \delta A_{\rm s}(t), \delta A_{\rm s}^\dagger(t), \delta P_1(t), \delta P_1^\dagger(t), \delta P_2(t), \delta P_2^\dagger(t), \delta P_r(t), \delta P_r^\dagger(t), \delta S_{z1}(t), \delta S_{z2}(t)]^{\rm T}. \quad (31)$$

To obtain equations for the fields and atomic fluctuations we linearize equations (24–25), and equations (26–30) and their Hermitian conjugates. We thus obtain a set of equations which can be written in a matrix form as:

$$\frac{{\rm d}\,|\delta\xi(t)]}{{\rm d}t} = -\,[B]\,|\delta\xi(t)] + |F_\xi(t)]\,. \quad (32)$$

$[B]$ is the linearized evolution matrix of the atom-field system and the column vector $|F_\xi(t)]$ contains the Langevin forces:

$$|F_\xi(t)] = |\sqrt{2\kappa/\tau}\delta A_{\rm p}^{\rm in}(t), \sqrt{2\kappa/\tau}\delta A_{\rm p}^{\rm in\dagger}(t), \sqrt{2\kappa/\tau}\delta A_{\rm s}^{\rm in}(t), \sqrt{2\kappa/\tau}\delta A_{\rm s}^{\rm in\dagger}(t), F_{P_1}(t), F_{P_1^\dagger}(t), F_{P_2}(t), F_{P_2^\dagger}(t), F_{P_r}(t), F_{P_r^\dagger}(t), F_{S_{z1}}(t), F_{S_{z2}}(t)]^{\rm T}. \quad (33)$$

The correlation matrix $[G(t)]$ of the fluctuations is:

$$[G(t)] = |\delta\xi(t)]\,[\delta\xi(0)|\,. \quad (34)$$

As before the variances are obtained from the zero time correlation functions, contained in matrix $[G(0)]$ which verifies:

$$[B]\,[G(0)] + [G(0)]\,[B]^\dagger = [D] \quad (35)$$

where $[D]$ is the correlation matrix of the Langevin forces:

$$|F_\xi(t)]\,[F_\xi(t')| = [D]\,\delta(t-t'). \quad (36)$$

$[G_{\rm c}(0)]$, the 4×4 lower diagonal block of $[G(0)]$, contains the variances of P_r, $P_r^\dagger$, S_{z1} and S_{z2}.

3 Spin squeezing

3.1 Definition

In the same way as a squeezed state of the electromagnetic field is defined by comparison to the coherent state, a squeezed spin state will be defined as having fluctuations in one component lower than the one of a coherent spin state [14]. A coherent spin state for N atoms is defined as an ensemble of N uncorrelated spins, each of them being an eigenstate with eigenvalue $+1/2$ of the individual spin operator in the (θ,ϕ) direction:

$$\sigma_{\theta,\phi i} = \sigma_{xi}\sin\theta\cos\phi + \sigma_{yi}\sin\theta\sin\phi + \sigma_{zi}\cos\theta \quad (37)$$

with $\sigma_{xi} = (\sigma_i + \sigma_i^\dagger)/2$, $\sigma_{yi} = (\sigma_i - \sigma_i^\dagger)/2{\rm i}$. This coherent spin state is an eigenvalue of the collective spin operator $S_{\theta,\phi} = \sum_{i=1,N}\sigma_{\theta,\phi i}$, with eigenvalue $S = N/2$ [25]. It satisfies the minimum uncertainty relationship with fluctuations equally distributed over any two orthogonal components normal to the (θ,ϕ)-direction, the variance of which is equal to $S/2 = N/4$. If one can squeeze the fluctuations of the total spin within the plane orthogonal to the mean value, it will result in noise reduction in spin measurements. The condition for spin squeezing is then [26]

$$\Delta S_\alpha \le \langle S_Z\rangle/2 \quad (38)$$

where the axes have been rotated in such a way that the Z-axis is in the direction of the mean spin and α represents a direction in the X,Y plane. $\langle S_Z\rangle$ is then the mean value of the spin and S_X and S_Y have zero mean values. This can only occur for a spin ensemble with $N > 1$ because it implies the emergence of quantum correlations within the spin ensemble.

We calculate the variances ΔS_X and ΔS_Y of the spin variables in the new reference frame. For this, we perform a rotation defined by angles ϕ around the z-axis and θ around the Y-axis (defined by the previous rotation) such that

$$\cos\theta = \langle S_z\rangle/s, \quad \cos\phi = \langle S_x\rangle/s_\phi \quad (39)$$

with $s = \sqrt{\langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2}$ and $s_\phi = \sqrt{\langle S_x\rangle^2 + \langle S_y\rangle^2}$, where the mean values are the solutions of the steady state equations.

A spin component in the (X,Y)-plane making an angle α with the X-axis has a variance given by:

$$\Delta S_\alpha = (\cos^2\alpha)\Delta S_X + (\sin^2\alpha)\Delta S_Y + (\sin 2\alpha){\rm Re}\,(\Delta S_{XY}) \quad (40)$$

with $\Delta S_{XY} = \langle\delta S_X(0)\ \delta S_Y(0)\rangle$. Then the values α_0 of α for the spin components having maximal and minimal variances verify:

$$\tan 2\alpha_0 = \frac{2\ {\rm Re}\,(\Delta S_{XY})}{\Delta S_X - \Delta S_Y}\,\cdot \quad (41)$$

In order to investigate squeezing, we compare the minimal variance to $\langle S_Z\rangle/2$. The corresponding normalized variance thus obtained is called $\Delta S_{\rm min}$, spin squeezing is achieved when $\Delta S_{\rm min} < 1$.

For the three-level system as mentioned, the spin fluctuations corresponding to the atomic coherence between levels 1 and 2 are given by lower diagonal block $[G_{\rm c}(0)]$ of the correlation matrix. To go into the relevant basis, one must first perform the transformation from P_r, $P_r^\dagger$, S_{z1}, S_{z2} to S_{cx}, S_{cy}, S_{cz} with:

$$S_{cx} = \frac{P_r + P_r^\dagger}{2}, \quad S_{cy} = \mathrm{i}\frac{P_r - P_r^\dagger}{2},$$
$$S_{cz} = \frac{\Pi_2 - \Pi_1}{2} = S_{z2} - S_{z1}. \tag{42}$$

For this we define matrix $[R]$:

$$[R] = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ \mathrm{i} & -\mathrm{i} & 0 & 0 \\ 0 & 0 & -2 & 2 \end{pmatrix}. \tag{43}$$

Then one goes into the basis S_{cX}, S_{cY}, where the mean spin direction is along OZ, by using the $[R_r]$ matrix:

$$[R_r] = \begin{pmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \tag{44}$$

with angles θ et φ given by equation (39)

The atomic correlation matrix $\left[G_c^\perp(0)\right]$ in basis S_{cX}, S_{cY} writes:

$$\left[G_c^\perp(0)\right] = [R_r]\,[R]\,[G_c(0)]\,[R]_{hc}\,[R_r]^T. \tag{45}$$

The variances in the XY-plane are then

$$\Delta S_{cX} = \left[G_c^\perp(0)\right]_{1,1}, \quad \Delta S_{cY} = \left[G_c^\perp(0)\right]_{2,2},$$
$$\Delta S_{cXY} = \left[G_c^\perp(0)\right]_{1,2}. \tag{46}$$

The minimal variance $\Delta S_{c\,min}$ corresponds to angle α_0 given by equation (41).

The correlation matrix $[G(0)]$ also allows us to compute the variances of the spins S_1 and S_2 associated with the coherences between levels 0–1 and 0–2. Spin S_1 involves operators P_1, $P_1^\dagger$, et S_{z1}. Using the order defined in equation (31) again we see that the concerned variances are given by the restriction of matrix $[G(0)]$ corresponding to ranks 5, 6 and 11. In the same way, for spin S_2, the variances are given by a restriction of $[G(0)]$ corresponding to ranks 7, 8 and 12. Reference frame changes similar to the one described above allows one to determine the minimal variances for these two spins.

In the case where levels 1 and 2 are Zeeman sublevels of an atomic state a, the spin squeezing can be detected by analyzing the polarization changes of a probe beam connecting state a to another state b different from level a. The optimal squeezing direction can be found by adjusting the polarization and propagation direction of the probe beam.

3.2 Coherent input field: self-spin squeezing

Let us first consider the case of a coherent input field for a two-level system. We have explored various sets of parameters. Spin squeezing has been found when the nonlinearity of the atomic ensemble is high, which also corresponds to conditions where the output field is squeezed.

Table 1. Minimal spin variances for a two-level system as a function of cooperativity C (the other parameters are adjusted for optimal noise reduction).

C	10	100	1 000	10 000
δ	0.8	4	15	42
I_c	0.9	2.6	8.9	23
ΔS_{min}	0.8	0.61	0.56	0.54

Table 2. Three-level system: minimal spin variances S_c et S_1 for $\delta_1 = 18$, $\beta_{p0} = 3.24$ and $C = 1\,000$ for a coherent input probe.

β_{s0}	0	0.1	0.2	0.22	0.25	0.3
δ_2		−0.5	−0.5	−0.5	−0.5	−0.5
δ_{c2}		−1	−1	−1	−1	0
$\Delta S_{1\,min}$	0.574	0.583	0.633	0.649	0.678	0.731
$\Delta S_{c\,min}$	1	0.926	0.848	0.847	0.854	0.89

Values for a few physical parameters are given in Table 1. Let us emphazise that the parameters chosen in Table 1 correspond to feasible experiments. For example in the experiments described in [7,9] using cesium atoms cooled in a magneto-optical trap, the dipole linewidth $\gamma/2\pi$ is equal to 2.6 MHz, and the optical cavity is such that $\kappa = 2\gamma$, the value of the incident field is then about 10 μW, and the cooperativity can reach 120. The value of the cooperativity could be increased using new trapping configurations to 1 000 and more [27].

It can be seen that spin squeezing increases with the cooperativity parameter and that squeezing values as high as 46% can be obtained. Let us mention that, for each value of the cooperativity C, the values of the intracavity laser intensity $I_c = \beta_0^2$, of the atomic detuning δ and of the cavity detuning δ_c have been optimized to obtain maximum squeezing. This squeezing is self-spin squeezing in as much it is due to the nonlinear action of the atomic ensemble on the light fluctuations inside the cavity, which yields squeezing in the collective atomic spin.

In the case of a three level system interacting with two coherent fields, we have concentrated on the configuration in which one field, the pump field, (interacting with transition 0–1) is a saturating one while the other one, the probe field, is far from saturation. Then, our calculations show that, for a given atomic number, $\Delta S_{c\,min}$ is smallest for parameters associated with transition 0–1 (atomic and cavity detunings, pump field intensity) close to the ones that minimize the noise on spin S_1 in the absence of the probe field. These parameters are given in Table 1.

We give the values of the variances of spins S_c and S_1 in Table 2 for a cooperativity $C = 1\,000$. As far as the parameters associated with transition $0 \to 2$ are concerned, the values of the scaled detunings δ_2 et δ_{c2} minimizing $\Delta S_{c\,min}$ depend very little on the probe amplitude β_{s0}.

When $\beta_{s0} = 0$, $\Delta S_{1\,min}$ takes the values ΔS_{min} given in Table 1. In the absence of the probe field, $\Delta S_{c\,min}$ is 1, which means that spin S_c is in a coherent state when the probe is the vacuum field. When β_{s0} is increased, $\Delta S_{c\,min}$

becomes smaller than 1 and the fluctuations of spin S_c are squeezed. At the same time, $\Delta S_{1\,\text{min}}$ increases: the squeezing of S_1 is degraded by the presence of the probe field. For higher values of the probe field intensity, the operating point for the pump field should be re-optimized in order to minimize $\Delta S_{c\,\text{min}}$.

3.3 Squeezed input field: squeezing transfer

For the two level system, we have examined the case in which the input light is a broadband squeezed vacuum. We have investigated the optimum conditions for this squeezing to be transferred to the atoms. The best conditions in this case correspond to strong coupling of the atomic ensemble with the cavity and to a very weak intracavity resonant field [28]. In contrast to the previous case, the atomic ensemble behaves like a linear system. We first study the case of exact resonance between the atoms and the cavity, which is the most favorable one for squeezing transfer. The amount of squeezing transferred to the atoms is always less than the one of the incoming light, because of the coupling of the atoms with the vacuum. The higher the squeezing of the incoming field, the better the spin squeezing. However, it should be noted that our model relying on linearization of the field at very low field intensities is not valid for high values of the squeezing (which would imply non negligible photon numbers). For a given value of the field squeezing, the amount of spin squeezing increases with the cooperativity C, and tends to a limit. This limit depends on the value of the ratio γ/κ, and the maximum efficiency of the squeezing transfer is equal to $\kappa/(\gamma+\kappa)$. This shows that the coupling of the atoms with the surrounding vacuum field limits the degree of achievable squeezing in the collective spin.

When the incoming field is not exactly resonant with both the atoms and the cavity, the situation is quite different. For large squeezing of the incoming light, excess noise is obtained for the atomic spin. The excess noise goes to infinity for perfect squeezing of the incoming light. This comes from the fact that non resonant atoms cause a rotation of the noise ellipse. As a result, the squeezed and anti-squeezed components are mixed inside the cavity, and induce excess noise on the spin.

We now turn to the three-level system and we consider the case in which the pump field is in a coherent state, while the probe field is a squeezed vacuum field. When the pump intensity is far from saturation that is the scaled pump amplitude β_{p0} smaller than 1, our model shows that the spin S_c associated with the 1–2 coherence is squeezed as is S_2 the spin associated with the 0–2 optical dipole interacting with the squeezed field. Moreover, $\Delta S_{c\,\text{min}}$ takes the same values as $\Delta S_{2\,\text{min}}$. We have seen above that the variance $\Delta S_{2\,\text{min}}$ is minimal when the optimal squeezing transfer condition $\delta_2 = \delta_{c2} = 0$ is fulfilled. The variance $\Delta S_{c\,\text{min}}$ is minimal in the same conditions. The choice of the pump detuning parameters δ_1 et δ_{c1} in non critical in this case ($\beta_{p0} \ll 1$).

In Table 3, we have again used parameters that correspond to feasible experiments $\gamma = 2.6$ MHz, $\kappa = 5.2$ MHz, $C = 1\,000$, $\delta_1 = \delta_{c1} = \delta_2 = \delta_{c2} = 0$. The probe field is assumed to be almost perfectly squeezed. When $\beta_{p0} = 0$ the limit value $\gamma/(\gamma+\kappa) = 1/3$ is found again for $\Delta S_{2\,\text{min}}$, but $\Delta S_{c\,\text{min}}$ is not defined since there is no population in levels 1 and 2. When the pump amplitude β_{p0} assumes a small non zero value, $\Delta S_{c\,\text{min}}$ et $\Delta S_{2\,\text{min}}$ are equal and equal to the optimal 1/3 value. When the pump is increased further, the noise reduction on S_c and S_2 is degraded by the high pump field. $\Delta S_{1\,\text{min}}$ can then become squeezed by a self-squeezing effect.

Table 3. Three-level system: minimal variances for the three spins S_c, S_1 et S_2 for a squeezed vacuum probe with all detunings equal to zero, $C = 1\,000$ et $\gamma/\kappa = 0.5$.

β_{p0}	0	0.1	0.3	0.5	0.7
$\Delta S_{1\,\text{min}}$	1	0.99	0.91	0.83	0.77
$\Delta S_{2\,\text{min}}$	0.34	0.34	0.35	0.38	0.4
$\Delta S_{c\,\text{min}}$		0.34	0.41	0.5	0.6

4 Conclusion

Using a full quantum model for ensembles of two- and three-level atoms in a cavity, we have derived the atomic spin fluctuation spectra and variances and we have shown rigorously the occurrence of spin squeezing in such systems. These results are likely to be generalized to atoms interacting with two fields in a Raman (Λ) type configuration.

Spin squeezing may occur in two different cases. In the first one, the non-linearity of the atomic ensemble is exploited to squeeze the intracavity field, which in turn imprints squeezing on the atomic ensemble, yielding self-spin squeezing. In the second one, the atomic ensemble has a linear behavior. It cannot create squeezing in the intracavity field. However, if one of the incoming fields is squeezed, the atom-field coupling in the cavity yields spin squeezing.

Appendix

We give here the expression of the diffusion matrix appearing in equation (15) for the two-level system. The matrix elements of the higher 2×2 quadrant of $[D]$ are the correlation functions of a broadband squeezed field, equal to the one of the single mode squeezed field defined in equations (16, 17)

$$\langle \delta A^{\text{in}}(t)\delta A^{\text{in}\dagger}(t)\rangle = \cosh^2(r)\delta(t-t'), \quad (47)$$

$$\langle \delta A^{\text{in}}(t)\delta A^{\text{in}}(t)\rangle = (1/2)\sinh(r)\mathrm{e}^{\mathrm{i}\theta}\delta(t-t'), \quad (48)$$

$$\langle \delta A^{\text{in}\dagger}(t)\delta A^{\text{in}\dagger}(t)\rangle = (1/2)\sinh(r)\mathrm{e}^{-\mathrm{i}\theta}\delta(t-t'), \quad (49)$$

$$\langle \delta A^{\text{in}\dagger}(t)\delta A^{\text{in}}(t)\rangle = \sinh^2(r)\delta(t-t'). \quad (50)$$

The matrix elements of the lower 3×3 quadrant of $[D]$ are the correlation functions of the atomic noise operators appearing in equations (9, 10). They were evaluated with the Einstein generalized relations [22]. The only non zero ones are given below:

$$\langle F_P(t)F_{P^\dagger}(t')\rangle = 2\gamma N\delta(t-t'), \quad (51)$$

$$\langle F_P(t)F_{S_z}(t')\rangle = 2\gamma p_0\delta(t-t'), \quad (52)$$

$$\langle F_{S_z}(t)F_{P^\dagger}(t')\rangle = 2\gamma p_0^*\delta(t-t'), \quad (53)$$

$$\langle F_{S_z}(t)F_{S_z}(t'))\rangle = 2\gamma(N/2+s_{z0})\delta(t-t'). \quad (54)$$

The other elements of $[D]$ are equal to zero since there is no correlations between atomic and fields fluctuations at the same time. So we get:

$$[D] = \begin{bmatrix} (2\kappa/\tau)\mathrm{ch}^2(r) & (\kappa/\tau)\mathrm{sh}(2r)\,\mathrm{e}^{\mathrm{i}\theta} & 0 & 0 & 0 \\ (\kappa/\tau)\mathrm{sh}(2r)\,\mathrm{e}^{-\mathrm{i}\theta} & (2\kappa/\tau)\mathrm{sh}^2(r) & 0 & 0 & 0 \\ 0 & 0 & 2\gamma N & 0 & 2\gamma p_0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\gamma p_{0c} & 0 & 2\gamma(N/2+s_{z0}) \end{bmatrix}.$$

As far as the three level system is concerned the diffusion matrix for the fields is

$$[D_{\mathrm{ch}}] = 2\kappa/\tau \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{ch}^2 r & \mathrm{sh}(2r)/2\mathrm{e}^{\mathrm{i}\theta} \\ 0 & 0 & \mathrm{sh}(2r)/2\mathrm{e}^{-\mathrm{i}\theta} & \mathrm{sh}^2 r \end{pmatrix}. \quad (55)$$

Since there are no correlations between incoming fields and atoms at the same time, $[D]$ can be written as:

$$[D] = \begin{pmatrix} [D_{\mathrm{ch}}] & 0 \\ 0 & [D_{\mathrm{at}}] \end{pmatrix} \quad (56)$$

where $[D_{\mathrm{at}}]$ is computed as previously from the generalized Einstein equations.

References

1. M. Fleischhauer, S.F. Yelin, M.D. Lukin, Opt. Commun. **179**, 395 (2000).
2. A. Kuzmich, E.S. Polzick, Phys. Rev. Lett. **85**, 5639 (2000).
3. E.A. Cornell, Nature **409**, 461 (2001).
4. M.D. Reid, Phys. Rev. A **37**, 4792 (1988).
5. F. Castelli, L.A. Lugiato, M. Vadacchino, Nuovo Cimento B **10**, 183 (1998).
6. L. Hilico, C. Fabre, S. Reynaud, E. Giacobino, Phys. Rev. A **46**, 4397 (1992).
7. A. Lambrecht, T. Coudreau, A.M. Steinberg, E. Giacobino, Europhys. Lett. **36**, 93 (1996).
8. A. Sinatra, J.F. Roch, K. Vigneron, Ph. Grelu, J.Ph. Poizat, K. Wang, P. Grangier, Phys. Rev. A **57**, 2980 (1998).
9. T. Coudreau, L. Vernac, A.Z. Khoury, G. Breitenbach, E. Giacobino, Europhys. Lett. **46**, 722 (1999).
10. D.F. Walls, P. Zoller, Phys. Rev. Lett. **47**, 709 (1981).
11. C.W. Gardiner, *Handbook of Stochastic Methods* (Springer Verlag, 1985); H.J. Carmichael, Phys. Rev. A **33**, 3262 (1986).
12. M. Kitagawa, M. Ueda, Phys. Rev. A **47**, 5138 (1993).
13. W.M. Itano, J.C. Bergquist, J.J. Bollinger, J.M. Gilligan, D.J. Heinzen, F.L. More, M.G. Raizen, D.J. Wineland, Phys. Rev. A **47**, 3554 (1993).
14. D.J. Wineland, J.J. Bollinger, W.M. Itano, D.J. Heinzen, Phys. Rev. A **50**, 67 (1994).
15. L. Vernac, M. Pinard, E. Giacobino, Phys. Rev. A **62**, 063812 (2000).
16. C.W. Gardiner, Phys. Rev. Lett. **56**, 1917 (1986).
17. J.M. Courty, S. Reynaud, Europhys. Lett. **10**, 237 (1989).
18. P.R. Rice, L.M. Pedrotti, J. Opt. Soc. Am. B **9**, 2008 (1992).
19. Q.Q. Turchette, N.Ph. Georgiades, C.J. Hood, H.J. Kimble, Phys. Rev. A **58**, 4056 (1998).
20. A. Kuzmich, K. Moelmer, E.S. Polzik, Phys. Rev. Lett. **79**, 4782 (1997).
21. J. Hald, J.L. Soerensen, C. Schori, E.S. Polzik, Phys. Rev. Lett. **83**, 1319 (1999).
22. C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, *Atom-Photons Interactions* (Wiley, New-York, 1991), p. 385.
23. M.O. Scully, M.S. Zubairy, *Quantum Optics* (Cambridge University Press, 1997), Ch. 2.
24. C. Fabre, S. Reynaud, “Quantum noise in optical systems: a semi classical approach”, Les Houches, session LIII, 1990, *Fundamental systems in Quantum Optics* (Elsevier Science Publishers B.V., 1992).
25. G.S. Agarwal, R.R. Puri, R.P. Singh, Phys. Rev. A **56**, 2249 (1997).
26. K. Wodkiewicz, J.H. Eberly, J. Opt. Soc. Am. B **2**, 458 (1985).
27. J. Soding, D. Guéry Odelin, P. Desbiolles, G. Ferrari, J. Dalibard, Phys. Rev. Lett. **80**, 1869 (1998).
28. L. Vernac, M. Pinard, E. Giacobino, Eur. Phys. J. D **17**, 125 (2001).